LENGTH SPECTRUM OF PERIODIC RAYS FOR BILLIARD FLOW

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Abstract. We study for several compact strictly convex disjoint obstacles the length spectrum $\mathcal{L}$ formed by the lengths of all primitive periodic reflecting rays. We prove the existence of sequences $\{\ell_j\}$, $\{m_j\}$ with $\ell_j \in \mathcal{L}$, $m_j \in \mathbb{N}$ such that the condition (LB) related to the dynamical zeta function $\eta_D(s)$ is satisfied. This condition implies the existence of lower bounds for the number of the scattering resonances for Dirichlet Laplacian. We construct such sequences under some separation condition for a small subset of $\mathcal{L}$ corresponding to lengths of the periodic rays with even reflexions. Our separation condition is weaker than the assumption of exponentially separated length spectrum $\mathcal{L}$. Moreover, we show that the periodic orbits in the phase space are exponentially separated.

Keywords: billiard flow, periodic reflecting rays, length spectrum, separation condition

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with C^∞ smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow φ_t in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary ∂D. We consider the (non-grazing) billiard flow φ_t (see [CP, Section 2.2], [Pet25b, Section 2] for the definition) and the periodic trajectories will be called periodic rays. For any periodic ray γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by $\mathcal{P}$ the set of all oriented periodic rays and by P_γ, $\gamma \in \mathcal{P}$, the associated linearised Poincaré map (see [PS17, Section 2.3] for the definition). Consider the

Dirichlet dynamical zeta function

$$\eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},\ \mathrm{Re}\, s \gg 1. \tag{1.2}$$

We have the estimates (see for instance [Pet99, Appendix])

$$C_1 e^{\mu_1 \tau(\gamma)} \leq |\det(\mathrm{Id} - P_\gamma)| \leq e^{\mu_2 \tau(\gamma)},\ \forall \gamma \in \mathcal{P} \tag{1.3}$$

with constants $C_1 > 0,\ 0 < \mu_1 < \mu_2$. The series $\eta_D(s)$ is absolutely convergent and not vanishing for sufficiently large $\mathrm{Re}\, s$.

The zeta function $\eta_D(s)$ is important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d \setminus D$ with Dirichlet boundary conditions on ∂D. For more details we refer to [Ika90b, Section 1], [CP, Section 1]. It was proved in [CP, Theorem 1 and Theorem 4] that η_D admits a *meromorphic continuation* to $\mathbb{C}$ with simple poles and integer residues. There is a conjecture that η_D cannot be prolonged as *entire function*. This conjecture was established for obstacles with real analytic boundary in [CP, Theorem 3] and for obstacles with sufficiently small diameters [Ika90b], [Sto09] and C^∞ smooth boundary.

The difficulties to examine the analytic singularities of $\eta_D(s)$ are related to the change of signs of the coefficients of the Dirichlet series (1.2) which may produce cancellations. To study these cancelations, introduce the distribution

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma) \delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}} \in \mathcal{S}'(\mathbb{R}^+).$$

Let $\psi \in C_0^\infty(\mathbb{R}; \mathbb{R}_+)$ be an even function with $\mathrm{supp}\, \psi \subset [-1, 1]$ such that $\psi(t) = 1$ for $|t| \leqslant 1/2$. Let $(\ell_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be sequences of positive numbers such that $\ell_j \to \infty$, $m_j \to \infty$ as $j \to \infty$ and

$$\ell_j \geqslant d_0 = 2 \min_{k \neq m} \mathrm{dist}\,(D_k, D_m) > 0,\ m_j \geqslant \max \big\{1, \frac{1}{d_0}\big\}.$$

Define

$$\psi_j(t) = \psi(m_j(t - \ell_j)),\ t \in \mathbb{R}.$$

Definition 1.1. We say that the condition (LB) for $F_D(t)$ is satisfied if there exist constants $\alpha_0 > 0, \alpha_1 > 0, c_1 > 0$ such that for all $\beta \geq \alpha_1$ there exist sequences $(\ell_j), (m_j)$ with $\ell_j \nearrow \infty$ as $j \to \infty$ and $e^{\beta \ell_j} \leqslant m_j \leqslant \mathrm{e}^{2\beta \ell_j}$ satisfying

$$|\langle \mathcal{F}_{\mathrm{D}}, \psi_j \rangle| \geqslant c_1 \mathrm{e}^{-\alpha_0 \ell_j},\ \forall j. \tag{1.4}$$

The estimate (1.4) gives exponentially small lower bounds for the sum of the contributions to $\langle F_D, \psi_j \rangle$ of the rays $\gamma \in \mathcal{P}$ with lengths

$$\tau(\gamma) \in (\ell_j - e^{-m_j}, \ell_j + e^{-m_j}),\ j \in \mathbb{N}.$$

If (LB) is satisfied, one obtains two important corollaries:

(i) The modified Lax-Phillips conjecture (MLPC) for scattering resonances introduced by Ikawa [Ika90a, page 212] holds. (MLPC) says that there exists a strip $\{z \in \mathbb{C} : 0 < \operatorname{Im} z \leq \alpha\}$ containing an infinite number of scattering resonances for Dirichlet Laplacian in $\mathbb{R}^d \setminus D$. For definition of scattering resonances and more precise results the reader may consult Chapter 5 in [LP89] for d odd and Chapter 4 in [DZ19]).

(ii) The function $\eta_D(s)$ has infinite number of poles in some strip $\{s \in \mathbb{C} : \operatorname{Re} s \geq \delta\}$ and we have a lower bound of the counting function of the poles in this strip (see [Pet25b, Theorem 1.1]). In fact, the result in [Pet25b, Theorem 1.1] has been stated assuming that $\eta_D(s)$ cannot be prolonged as an entire function, however the proof works if sequences (ℓ_j), (m_j) satisfying (1.4) exist.

On the other hand, Ikawa [Ika90a, Proposition 2.3] showed that if $\eta_D(s)$ cannot be prolonged as entire function, then (LB) holds for F_D. For obstacles with C^∞ boundary some conditions which imply that $\eta_D(s)$ cannot be prolonged as entire function have been established in [Pet25a]. It is interesting to find conditions leading to (LB) which are not related to the existence of poles of $\eta_D(s)$. In this paper we study this problem.

To construct sequences $\{\ell_j\}, \{m_j\}$ satisfying (1.4), we must study the distribution of the periods of periodic rays which has independent interest. Let $\Pi \subset \mathcal{P}$ be the set of primitive periodic orbits of billiard flow φ_t and let $\Pi_+ \subset \Pi$ (resp. $\Pi_- \subset \Pi$) be the set of periodic rays with even (resp. odd) number of reflexions. The counting function of the lengths satisfies

$$\sharp\{\gamma \in \Pi :\ \tau(\gamma) \leqslant x\} \sim \frac{e^{hx}}{hx}, \quad x \to +\infty, \tag{1.5}$$

with some $h > 0$ (see for instance, [PP90, Theorem 6.9] for weak-mixing suspension symbolic flows and [Ika90a], [Mor91] for symbolic models related to billiard flow). Moreover, we have the asymptotics (see [Gio10, Theorem 2])

$$\sharp\{\gamma \in \Pi_\pm :\ \tau(\gamma) \leqslant x\} \sim \frac{e^{hx}}{2hx}, \quad x \to +\infty. \tag{1.6}$$

Introduce the *length spectrum* $\mathcal{L} = \{\tau(\gamma) : \gamma \in \Pi\}$. We say that $\mathcal{L}$ is *exponentially separated* if there exists $\nu > 0$ such that for all $\ell,\ \ell' \in \mathcal{L}$

we have

$$|\ell - \ell'| \geq e^{-\nu \max\{\ell, \ell'\}} \text{ if } \ell \neq \ell'. \tag{1.7}$$

From Theorem 1.1 below it follows that if $\mathcal{L}$ is exponentially separated, then the condition (LB) holds.

We recall some positive and negative results concerning the exponential separation of length spectrum $\mathcal{L}$. For compact Riemannian manifolds M with negative curvatures the metrics for which $\mathcal{L}$ is not exponentially separated are topologically generic and dense for C^k, $k > 3$, topology (see [DJ16, Theorem 4.1]). On the other hand, Schenck proved in [Sch20, Theorem 1]) that the set of metrics for which $\mathcal{L}$ is exponentially separated is dense in $C^k, k \geq 2$, topology and (1.7) holds with $\nu = \nu_k > 0$ depending of k and the dynamical characteristic. However, $\nu_k \to +\infty$ as $k \to \infty$, so an approximation with C^∞ metrics having exponentially separated length spectrum is an open problem.

For billiard flow φ_t the lengths $\ell \in \mathcal{L}$ are rationally independent for generic obstacles (see [PS17, Theorem 6.2.3]). This result implies that generically there are gaps between the lengths of different periodic rays. However the estimates of these gaps and the existence of generic obstacles with exponentially separated $\mathcal{L}$ seems to be difficult open problem. In contrast to the metric case mentioned above, for obstacles we may perturb generically only the boundary and the rays in $\mathbb{R}^d \backslash \bar{D}$ are always union of linear segments. Consequently, a perturbation of the boundary is much more restrictive than the perturbations of a metric studied in [DJ16] and [Sch20]. In section 4 we prove that the periodic orbits in the phase space are exponentially separated. This is an analog of Proposition 2 in [Sch20]. This result could be considered as a first step in the analysis of the existence of exponentially small gaps in $\mathcal{L}$ for generic obstacles.

It is important to remark that in (1.4) are involved the contributions of the iterated rays with periods in the set $\{m\ell : \ell \in \mathcal{L}, m \geq 2\}$. Hence even in the case when $\mathcal{L}$ is exponentially separated, for the analysis of (LB) the terms in (1.4) related to these rays must be estimated. In this paper we show that a separation condition concerning a very small subsets of rays $\gamma \in \Pi_+$ implies (LB). Our main result is the following

Theorem 1.1. *Assume that there exist* $\delta > 0, 0 < \rho < \min\{1, h^{-1}\}, c_0 > 5 - \frac{h\rho}{3}$ *and a sequence* $q_j \nearrow +\infty$ *such that*

$$\sharp\Big\{\gamma \in \Pi_+ : \, q_j - \rho < \tau(\gamma) \leq q_j,$$
$$|\tau(\gamma) - \tau(\gamma')| \geq e^{-\delta \max\{\tau(\gamma), \tau(\gamma')\}}, \, \forall \gamma' \in \Pi \setminus \{\gamma\}\Big\} \geq \frac{c_0 \rho e^{\frac{hq_j}{3}}}{8q_j}. \tag{1.8}$$

Then the condition (LB) is satisfied for F_D.

In Lemma 3.1 we prove that for every small $\epsilon > 0$ and $q_j \geq C(\epsilon)$ we have the lower bound

$$\sharp\{\gamma \in \Pi_+ : \ q_j - \rho < \tau(\gamma) \leq q_j\} \geq (1-\epsilon)\frac{\rho e^{hq_j}}{8q_j},$$

while the separation assumption in (1.8) concerns only $\mathcal{O}\Big(\frac{\rho e^{\frac{hq_j}{3}}}{8q_j}\Big)$ rays. For this reason we say that a very small subsets of $\{\gamma \in \Pi_+ : \ q_j - \rho < \tau(\gamma) \leq q_j\}$ must be exponentially separated. Moreover, in Theorem 1.1 there is not separation condition for the lengths of $\gamma \in \Pi_-$.

The paper is organised as follows. In Section 2 we obtain upper and lower bounds of the number of iterated rays with odd and even number of reflexions. These bounds have independent interest. In particular, we show that the number of the iterated periodic rays with lengths in $[d_0/2, q]$ is less than the number of primitive periodic rays with lengths in the same interval. In Section 3 one examines the number of lengths of periodic rays in small intervals $]q_j - \rho, q_j]$ and we prove Theorem 1.1. The exponential separation of periodic rays in phase space is studied in Section 4. The idea of the proof is based on the fact that different periodic rays follows different configurations (see [PS17, Corollary 2.2.4]). The analysis is technical since we must study some rays having tangent segments. Finally, in Section 5 we formulate an open problem for generic obstacles.

2. Estimation of the number of iterated rays

Clearly, $d_0 \leq \tau(\gamma)$, $\forall \gamma \in \mathcal{P}$. Given $q \gg 1$, introduce the counting functions of the periods of iterated rays

$$N_{odd}(q) = \sharp\{\gamma \in \Pi_- : \ (2k+1)\tau(\gamma) \leq q, \ k \in \mathbb{N}, \ k \geq 1\},$$
$$N_{even}(q) = \sharp\{\gamma \in \Pi : \ 2k\tau(\gamma) \leq q, \ k \in \mathbb{N}, \ k \geq 1\}.$$

Therefore for $q \geq 4d_0$

$$(2k+1)d_0 \leq (2k+1)\tau(\gamma) \leq q \tag{2.1}$$

implies $k \leq [\frac{q}{2d_0} - 1/2] = p_q$, $p_q \geq 1$. Thus in the definition of $N_{odd}(q)$ one has $1 \leq k \leq p_q$, while in $N_{even}(q)$ we have $1 \leq k \leq [\frac{q}{2d_0}]$. If $\gamma \in \Pi_-$, the number of reflexions $m(\gamma)$ of γ is odd and the iterated ray

$$\gamma_{2k+1} = \underbrace{\gamma \cup \gamma \cup \ldots \cup \gamma}_{(2k+1)\ times}$$

with length $(2k+1)\tau(\gamma)$ will have odd reflexions, too. Hence the contribution of γ_{2k+1} in (1.2) contains a negative factor $(-1)^{(2k+1)m(\gamma)}$.

Proposition 2.1. *Let $0 < \epsilon < 1/4$ be fixed. Then there exists $B_\epsilon \gg 1$ such that for $q \geq B_\epsilon$ we have*

$$(1-\epsilon)\frac{3e^{\frac{hq}{3}}}{2hd} < N_{odd}(q) \leq (1+\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq}, \tag{2.2}$$

$$(1-\epsilon)\frac{2e^{\frac{hq}{2}}}{hd} < N_{even}(q) \leq (1+\epsilon)\frac{2e^{\frac{hq}{2}}}{hq}. \tag{2.3}$$

Proof. Write

$$N_{odd}(q) = \sum_{k=1}^{p_q} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\}.$$

Applying (1.6), there exists $C_\epsilon > d_0 + 1$ such that for $x \geq C_\epsilon$ we have

$$(1 - \frac{\epsilon}{2})\frac{e^{hx}}{2hx} \leq \sharp\{\gamma \in \Pi_\pm : \tau(\gamma) \leq x\} \leq (1 + \frac{\epsilon}{2})\frac{e^{hx}}{2hx}. \tag{2.4}$$

We fix C_ϵ and choose $q \geq B_\epsilon > \max\{5C_\epsilon, 4d_0\}$. We have the sum

$$N_{odd}(q) = \sum_{[\frac{q}{C_\epsilon}] \geq 2k+1 \geq 3} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\}$$

$$+ \sum_{[\frac{q}{C_\epsilon}] < 2k+1 \leq \frac{q}{d_0}} \sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq \frac{q}{2k+1}\Big\} = J_1(q) + J_2(q).$$

There exists a constant $A_\epsilon > 1$ such that

$$\sharp\Big\{\gamma \in \Pi_- : \tau(\gamma) \leq C_\epsilon\Big\} \leq A_\epsilon.$$

According to (2.1) and (2.4), one deduces

$$J_1(q) \leq (1 + \frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{h}\Big(\frac{1}{2q} + \frac{1}{3}\sum_{k=2}^{m(\epsilon,q)} \frac{e^{\frac{hq}{2k+1} - \frac{hq}{3}}}{d_0}\Big)$$

$$\leq (1 + \frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{h}\Big(\frac{1}{2q} + \frac{m(\epsilon,q) - 1}{3d_0}e^{-\frac{2hq}{15}}\Big),$$

where

$$m(\epsilon, q) = \begin{cases} [\frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2] \text{ if } \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \notin \mathbb{N}, \\ \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \text{ if } \frac{1}{2}[\frac{q}{C_\epsilon}] - 1/2 \in \mathbb{N}. \end{cases}$$

Notice that $q \geq 5C_\epsilon$ implies $m(\epsilon, q) \geq 2$. Since in $J_2(q)$ one has $2k+1 \geq [\frac{q}{C_\epsilon}] + 1 > \frac{q}{C_\epsilon}$, we obtain

$$J_2(q) \leq (p_q - m(\epsilon, q))A_\epsilon.$$

Increasing B_ϵ, if it is necessary, one arranges for $q \geq B_\epsilon$ the inequalities

$$\frac{1}{2q} + \frac{m(\epsilon, q) - 1}{3d_0} e^{-\frac{2hq}{15}} \leq \frac{1}{2q} + \frac{\epsilon}{8q(1+\epsilon/2)} = \frac{4+3\epsilon}{8q(1+\epsilon/2)},$$

$$(p_q - m(\epsilon, q))A_\epsilon \leq \frac{3\epsilon e^{\frac{hq}{3}}}{8hq}.$$

Combining the above estimates for $J_k(q),\ k = 1, 2$, we conclude that

$$N_{odd}(q) \leq \frac{(1+\epsilon)3e^{\frac{hq}{3}}}{2hq}.$$

To obtain the left hand side part of (2.2), we apply (2.4) and taking into account only the term

$$\sharp\{\gamma \in \Pi_- : \tau(\gamma) \leq q/3\},$$

one has

$$(1-\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq} < (1 - \frac{\epsilon}{2})\frac{3e^{\frac{hq}{3}}}{2hq} \leq N_{odd}(q).$$

For the proof of (2.3) we apply a similar argument and we omit the details. □

3. Length spectrum in small intervals

To estimate the number of periodic rays in Π_+ with lengths in a interval $(q - \rho, q]$, we need the following

Lemma 3.1. *Let $0 < \rho < \min\{1, h^{-1}\}$ and let $0 < \frac{2\epsilon}{1-\epsilon} \leq \frac{\rho h}{4}$. Then for $q \geq C(\epsilon)$ we have*

$$(1-\epsilon)\frac{\rho e^{hq}}{8q} \leq \sharp\{\gamma \in \Pi_+ : q-\rho < \tau(\gamma) \leq q\} \leq (5-h\rho)(1+\epsilon)\frac{\rho e^{hq}}{8q}. \quad (3.1)$$

Proof. An application of (1.6) with $q \geq C(\epsilon)$ yields

$$\sharp\{\gamma \in \Pi_+ :\ q - \rho < \tau(\gamma) \leq q\} \geq (1-\epsilon)\frac{e^{hq}}{2hq} - (1+\epsilon)\frac{e^{h(q-\rho)}}{2h(q-\rho)}$$

$$= (1-\epsilon)\frac{e^{hq}}{2hqe^{h\rho}}\Big(e^{h\rho} - \frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)}\Big).$$

Next, choosing $C(\epsilon)$ large enough, we obtain

$$\frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)} = \Big(1 + \frac{2\epsilon}{1-\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)$$

$$\leq 1 + \frac{\rho h}{4} + \frac{\rho}{q-\rho}\Big(1 + \frac{2\epsilon}{1-\epsilon}\Big) \leq 1 + \frac{\rho h}{4} + \frac{\rho^2 h^2}{32} < e^{\frac{h\rho}{4}}.$$

This implies

$$e^{h\rho} - \frac{(1+\epsilon)q}{(1-\epsilon)(q-\rho)} > e^{h\rho}(1 - e^{-\frac{3h\rho}{4}}).$$

On the other hand, we have the inequality $f(y) = 1 - e^{-3y} - y \geq 0$ for $0 \leq y \leq \frac{\log 3}{3}$ because

$$f'(y) \geq 0 \text{ for } 0 \leq y \leq \frac{\log 3}{3}.$$

Therefore $\rho < \frac{1}{h} < \frac{4 \log 3}{3h}$ yields $\frac{h\rho}{4} < \frac{\log 3}{3}$, hence

$$1 - e^{-\frac{3h\rho}{4}} \geq \frac{h\rho}{4},$$

and we obtain the left hand side of (3.1).

To establish the upper bound in (3.1), notice that for $q \geq C(\epsilon)$ one has

$$\sharp\{\gamma \in \Pi_+ :\ q - \rho \leq \tau(\gamma) \leq q\} \leq (1+\epsilon)\frac{e^{hq}}{2hq} - (1-\epsilon)\frac{e^{h(q-\rho)}}{2h(q-\rho)}$$

$$= (1+\epsilon)\frac{e^{hq}}{2hq}\Big(1 - \Big(1 - \frac{2\epsilon}{1+\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)e^{-h\rho}\Big).$$

Since $e^{-x} \geq 1 - x$ for $x \geq 0$, and $\frac{2\epsilon}{1+\epsilon} < \frac{h\rho}{4}$, we obtain

$$1 - \Big(1 - \frac{2\epsilon}{1+\epsilon}\Big)\Big(1 + \frac{\rho}{q-\rho}\Big)e^{-h\rho} \leq 1 - (1 - \frac{h\rho}{4})(1 - h\rho)$$

$$= h\rho\Big(\frac{5 - h\rho}{4}\Big).$$

This completes the proof. □

It is important to note that in the estimates (3.1) one has as factor the length of the interval $[q-\rho, q]$. Introduce

$$N_{odd}(q-\rho, q) = N_{odd}(q) - N_{odd}(q-\rho).$$

Clearly, $h\rho < 1$ implies $h\rho/3 < 1$. Exploiting (2.2), we obtain the following

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *for* $q \geq C_\epsilon$ *we have*

$$(1-\epsilon)\frac{\rho e^{\frac{hq}{3}}}{8q} \leq N_{odd}(q-\rho, q) \leq (5 - \frac{h\rho}{3})(1+\epsilon)\frac{\rho e^{\frac{hq}{3}}}{8q}. \tag{3.2}$$

We apply (2.2) and get

$$N_{odd}(q) - N_{odd}(q-\rho) \le (1+\epsilon)\frac{3e^{\frac{hq}{3}}}{2hq} - (1-\epsilon)\frac{3e^{\frac{h}{3}(q-\rho)}}{2h(q-\rho)}.$$

Next the proof is a repetition of that of Lemma 3.1 and we omit the details.

Proof of Theorem 1.1. First we choose $0 < \epsilon < 1$ small enough to arrange $c_0 > \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)$, $\frac{2\epsilon}{1-\epsilon} < \frac{rh}{4}$. Fix ϵ and consider the interval

$$(q_j - \rho - e^{-\delta q_j}, q_j + e^{-\delta q_j}] = (p_j - \rho_j, p_j]$$

with $p_j = q_j + e^{-\delta q_j}$ and $\rho_j = \rho + 2e^{-\delta q_j}$. Taking q_j large enough, one gets $\rho_j < \min\{1, h^{-1}\}$. We apply the upper bound in (3.2) for $N_{odd}(p_j - \rho_j, p_j)$ with $q_j \ge C(\epsilon)$ and deduce

$$N_{odd}(p_j - \rho_j, p_j) \le \frac{15 - h\rho_j}{3}(1+\epsilon)\frac{\rho_j e^{\frac{hp_j}{3}}}{8p_j}. \tag{3.3}$$

We claim that for $q_j \ge m(\epsilon) \ge C(\epsilon)$ large we have

$$(15 - h\rho_j)\frac{\rho_j e^{\frac{h}{3}e^{-\delta q_j}}}{p_j} < (15 - h\rho + \epsilon)\frac{\rho}{q_j}. \tag{3.4}$$

This inequality is equivalent to

$$\Big(1 - \frac{e^{-\delta q_j}}{p_j}\Big)\Big(1 + \frac{2e^{-\delta q_j}}{\rho}\Big)e^{\frac{h}{3}e^{-\delta q_j}} < 1 + \frac{2he^{-\delta q_j} + \epsilon}{15 - h\rho_j}.$$

For $q_j \to +\infty$ the left hand side of the above inequality goes to 1, so for large q_j it will be less than $1 + \frac{\epsilon}{15 - h\rho} < 1 + \frac{\epsilon}{15 - h\rho_j}$. This proves the claim. Consequently, for $q_j \ge m(\epsilon)$ the estimate (3.4) implies

$$N_{odd}(p_j - \rho_j, p_j) \le \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)\frac{\rho e^{\frac{hq_j}{3}}}{8q_j}.$$

Increasing $m(\epsilon)$ and taking into account (1.8), for $q_j \ge m(\epsilon)$ we obtain

$$\sharp\Big\{\gamma \in \Pi_+ : q_j - \rho < \tau(\gamma) \le q_j, |\tau(\gamma) - \tau(\gamma')| \ge e^{-\delta \max\{\tau(\gamma), \tau(\gamma')\}}, \forall \gamma' \in \Pi \backslash \{\gamma\}\Big\}$$

$$\ge \frac{c_0 \rho e^{\frac{hq_j}{3}}}{8q_j} > \frac{1}{3}(15 - h\rho + \epsilon)(1+\epsilon)\frac{\rho e^{\frac{hq_j}{3}}}{8q_j} \ge N_{odd}(p_j - \rho_j, p_j).$$

This means that the number of rays $\gamma \in \Pi_+$ with $q_j - \rho < \tau(\gamma) \le q_j$ such that the intervals

$$J_{\delta,j}(\gamma) = (\tau(\gamma) - e^{-\delta q_j}, \tau(\gamma) + e^{-\delta q_j})$$

contain only one $\tau(\gamma)$ with $\gamma \in \Pi$ is greater than $N_{odd}(p_j - \rho_j, p_j)$. Hence there exists $\gamma_j \in \Pi_+$ with $q_j - \rho < \tau(\gamma_j) \leq q_j$ such that $J_{\delta,j}(\gamma_j)$ does not contain the lengths of periodic rays $\gamma' \in \mathcal{P} \setminus \Pi$ having odd number of reflexions. On the other hand, some lengths of iterated rays with even number of reflexions could be in the interval $J_{\delta,j}(\gamma_j)$.

We choose $\ell_j = \tau(\gamma_j)$, $\beta = \delta$, $m_j = e^{\delta \ell_j}$. Then in the interval $L_j = (\ell_j - m_j^{-1}, \ell_j + m_j^{-1})$ we have only lengths of periodic rays with even number of reflections and $\psi_j(\ell_j) = 1$. By using (1.3), we conclude that

$$\langle F_D, \psi_j \rangle = \sum_{\tau(\gamma) \in L_j} \tau^\sharp(\gamma) |\det(\mathrm{Id} - P_\gamma)|^{-1/2} \psi_j(\tau(\gamma)) \geq d_0 e^{-\frac{\mu_2}{2}\ell_j}.$$

This completes the proof of Theorem 1.1.

4. Separation of periodic orbits in phase space

We start with some preparations. Let $ch(U)$ denote the convex hull of $U \subset \mathbb{R}^d$. For $k = 1, ..., r$, define

$$\epsilon_k = \mathrm{dist}\,\Big(ch(\bigcup_{k \neq j} D_j), D_k\Big).$$

Set

$$d_1 = \max_{k \neq j} \mathrm{dist}\,(D_k, D_j),\; d_2 = \frac{2d_1}{d_0} \geq 1.$$

The condition (1.1) implies $\epsilon_k \neq 0$, hence $\eta_0 > 0$.

We recall some notations concerning billiard flow φ_t (see for more details [CP, Section 2]). Let $S\mathbb{R}^d$ be the unit tangent bundle of $\mathbb{R}^d$ and let $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ be the natural projection. For $x \in \partial D_j$, denote by $n_j(x)$ the inward unit normal vector to ∂D_j at x pointing into D_j. Set $D = \bigcup_{j=1}^r D_j$ and

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d \;:\; x \in \partial D\}.$$

Define the grazing set $\mathcal{D}_{\mathrm{g}} = T(\partial D) \cap \mathcal{D}$ and denote by $\langle ., . \rangle$ the scalar product in $\mathbb{R}^d$. We say that $(x, v) \in T_{\partial D_j}\mathbb{R}^d$ is incoming (resp. outgoing) if $\langle v, n_j(x) \rangle > 0$ (resp. $\langle v, n_j(x) \rangle < 0$). Introduce

$$\mathcal{D}_{\mathrm{in}} = \{(x, v) \in \mathcal{D} \;:\; (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\mathrm{out}} = \{(x, v) \in \mathcal{D} \;:\; (x, v) \text{ is outgoing}\}.$$

For $(x, v) \in \mathcal{D}_{\mathrm{in/out/g}}$ denote by $v' \in \mathcal{D}_{\mathrm{out/in/g}}$ the image of v by the reflexion R_x with respect to $T_x(\partial D)$ at $x \in \partial D$, that is

$$v' = v - 2\langle v, n_j(x) \rangle n_j(x), \quad v \in S_x\mathbb{R}^d, \quad x \in \partial D_j.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is a complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x,v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ we set

$$\tau_\pm(x,v) = \pm \inf\{t \geqslant 0 : x \pm tv \in \partial D\}.$$

By convention, we have $\tau_\pm(x,v) = \pm\infty$, if the ray $x \pm tv$ has no common point with ∂D for $\pm t > 0$. For $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{g}}$ we define

$$\phi_t(x,v) = (x+tv, v), \quad t \in [\tau_-(x,v), \tau_+(x,v)],$$

while for $(x,v) \in \mathcal{D}_{\mathrm{in/out}}$, we set

$$\phi_t(x,v) = (x+tv,v) \quad \text{ if } \quad \begin{cases} (x,v) \in \mathcal{D}_{\mathrm{out}},\ t \in [0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\mathrm{in}},\ t \in [\tau_-(x,v), 0]\,, \end{cases}$$

and

$$\phi_t(x,v) = (x+tv', v') \quad \text{ if } \quad \begin{cases} (x,v) \in \mathcal{D}_{\mathrm{in}},\ t \in \,]0, \tau_+(x,v)]\,, \\ \text{or } (x,v) \in \mathcal{D}_{\mathrm{out}},\ t \in [\tau_-(x,v'), 0[\,. \end{cases}$$

Introduce the non-grazing billiard table M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right),$$

where $(x,v) \sim (y,w)$ if and only if $(x,v) = (y,w)$ or

$$x = y \in \partial D \quad \text{ and } \quad w = v'.$$

The set M is endowed with the quotient topology.

The non-grazing flow φ_t is defined on M as follows. For $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{in}}$ we set

$$\varphi_t([(x,v)]) = [\phi_t(x,v)], \quad t \in \,]\tau_-^{\mathrm{g}}(x,v), \tau_+^{\mathrm{g}}(x,v)[\,,$$

where $[z]$ denotes the equivalence class of $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\mathrm{g}}(x,v) = \pm \inf\{t > 0 : \phi_{\pm t}(x,v) \in \mathcal{D}_{\mathrm{g}}\}.$$

Notice that $\tau_\pm^{\mathrm{g}}(x,v) \neq 0$ for $(x,v) \in \mathcal{D}_{\mathrm{in}}$, while it is possible to have $\tau_\pm^{\mathrm{g}}(x,v) = \pm\infty$. The above formula defines a flow on M since each $(x,v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})) \cup \mathcal{D}_{\mathrm{in}}$. Therefore φ_t is continuous, but the flow trajectory of the point (x,v) for times $t \notin \,]\tau_-^{\mathrm{g}}(x,v), \tau_+^{\mathrm{g}}(x,v)[$ is not defined. The flow φ_t is defined for all $t \in \mathbb{R}$ for z in the trapping set K formed by points $z \in M$ such that $-\tau_-^{\mathrm{g}}(z) = \tau_+^{\mathrm{g}}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty, \text{ when } A(z) = \{t \in \mathbb{R} : \pi(\varphi_t(z)) \in \partial D\}.$$

(for more details see [CP, Section 2]). It is easy to see that the condition (1.1) implies the existence of $\psi_0 \in (\pi/2, \pi)$ with the following property: if three points x, y, z belong to ∂D_{i_1}, ∂D_{i_2}, ∂D_{i_3}, $i_1 \neq i_2$, $i_2 \neq i_3$, respectively, the open segments (x, y) and (y, z) have no common points with D and $[x, y]$ and $[y, z]$ satisfy the reflection law at y, then $\psi > \psi_0$, where $\psi \in (\pi/2, \pi)$ is the angle between $[y, z]$ and the normal $n_{i_2}(y)$ of ∂D_{i_2} at y. Introduce

$$\mathcal{D}_{\rm out,0} = \{(x, v) \in \partial D \times \mathbb{S}^{d-1} :\ \langle v, n(x)\rangle \leq \cos\psi_0 < 0\}$$

and define the *billiard ball map*

$$\mathbf{B} : \mathcal{D}_{\rm out,0} \ni (x, v) \longmapsto (y, w) \in \overline{\mathcal{D}}_{\rm out},$$

where

$$(y, w) = (x + \tau_+^{\rm g}(x, v)v, R_x v),$$

and $R_x : v \in S_x\mathbb{R}^d \to v' \in S_x\mathbb{R}^d$ is called *reflection map.* The map $\mathbf{B}(x, v)$ is defined if $\tau_+^{\rm g}(x, v) < +\infty$.

Consider a point $\rho = (x, v) \in \mathring{B}$. Assume that $\phi_t(\rho)$ is reflecting ray with $p \geq 1$ reflexions starting at $\rho \in \mathring{B}$ for $t = 0$ and going to

$$\phi_t(\rho) = \Big(\phi_\sigma \circ \mathbf{B}^p \circ R \circ \phi_\tau\Big)(\rho) \in \mathring{B},\ t > 0,\ p \geq 1,\ \tau > 0,\ \sigma > 0 \quad (4.1)$$

with $R \circ \phi_\tau(\rho) \in \mathcal{D}_{\rm out,0}$,where

$$R :\ (y, w) \in \mathcal{D}_{\rm in} \to (y, R_y w) \in \mathcal{D}_{\rm out}.$$

The map $\mathbf{B} : \mathcal{D}_{\rm out,0} \to \mathcal{D}_{\rm out,0}$ is C^∞ smooth and

$$\|{\rm d}\mathbf{B}\|_{T(\partial D)\to T(\partial D)} \leq A_0$$

with constant $A_0 > 1$ depending of d_1, ψ_0 and the sectional curvatures of ∂D (see for instance, [CP, Appendix A]). On the other hand, $R \circ \phi_\tau$ is also C^∞ smooth and we have the diagram

$$\begin{array}{ccc} \mathring{B} & \xrightarrow{\ \phi_t\ } & \mathring{B} \\ \Big\downarrow {\scriptstyle R\circ\phi_\tau} & & \Big\uparrow {\scriptstyle \phi_\sigma} \\ \mathcal{D}_{\rm out,0} & \xrightarrow{\ \mathbf{B}^p\ } & \mathcal{D}_{\rm out,0} \end{array}$$

Consequently, ${\rm d}\phi_t = {\rm d}\phi_\sigma \circ {\rm d}\mathbf{B}^p \circ {\rm d}R \circ {\rm d}\phi_\tau$. Setting $\beta = 2\log A_0/d_0$, one deduces

$$\|{\rm d}\mathbf{B}^p\| \leq A_0^p = e^{\beta p d_0/2} < e^{\beta t},$$

where $t > pd_0/2$ is the length of γ and we obtain the estimate

$$\|{\rm d}\phi_t(\rho)\|_{T(\mathring{B})\to T(\mathring{B})} \leq C_0 e^{\beta t} \quad (4.2)$$

with $C_0 \geq 1,\ \beta > 0$ independent of ρ, τ, σ and p. Here $\|.\|$ is the norm induced on $T(\mathring{B})$ by the standard norm in $S\mathbb{R}^d$.

Every periodic reflecting ray γ is determined by a configuration

$$\alpha_\gamma = (i_1, \dots, i_k),$$

where $i_j \in \{1, \dots, r\}$, with $i_k \neq i_1,\ i_j \neq i_{j+1}$ for $j = 1, ..., k-1$ and α_γ is such that γ has *successive reflections* on $\partial D_{i_1}, \dots, \partial D_{i_k}$. The configuration α_γ is well defined modulo cyclic permutation. We say that γ has type α_γ and α_γ has length k. Moreover, according to [PS17, Corollary 2.2.4], for a fixed configuration α_γ there exists at most one periodic ray γ in $\mathbb{R}^d \setminus \mathring{D}$.

Given a periodic ray γ in $\mathbb{R}^d \setminus \mathring{D}$, define by $\tilde{\gamma}$ one of the two possible lifts

$$\tilde{\gamma}(t) = \{\varphi_t(x, \pm v) \in M :\ 0 \leq t < \tau(\gamma),\ x \in \gamma,\ x \notin \partial D\}$$

on M, where $v \in \mathbf{S}^{d-1}$ is the direction of γ at x. Below we fix a lift $\tilde{\gamma} = \tilde{\gamma}(t)$ corresponding to (x, v) and parametrised by the length. We will say that $\tilde{\gamma}(t)$ follows a configuration α, if $\pi(\tilde{\gamma}(t))$ follows α. Set

$$\mathcal{G}(T) = \{\tilde{\gamma} :\ \pi(\tilde{\gamma}) = \gamma \in \Pi,\ \tau(\gamma) \leq T\}.$$

A point $z \in \mathring{B}$ will be called *linearly connected* to $\tilde{\gamma}$ if there exists $w \in \tilde{\gamma} \cap \mathring{B}$ such that $\sigma z + (1-\sigma)w \in \mathring{B},\ \forall \sigma \in [0, 1]$. For such points $z \in \mathring{B}$ define

$$\mathrm{dist}(z, \tilde{\gamma}) = \min\{\|z - w\| :\ w \in \tilde{\gamma} \cap \mathring{B}, \sigma z + (1-\sigma)w \in \mathring{B},\ \forall \sigma \in [0, 1]\},$$

$$\Theta_{\tilde{\gamma}}^{\epsilon} = \{z \in \mathring{B} :\ z \text{ is linearly connected to } \tilde{\gamma},\ \mathrm{dist}(z, \tilde{\gamma}) \leq \epsilon\}.$$

We will prove the following result.

Theorem 4.1. *There exists $\epsilon_0 > 0$ depending of C_0, d_0 and η_0 such that for any different periodic rays $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathcal{G}(T)$ we have*

$$\Theta_{\tilde{\gamma}_1}^{\epsilon_0 e^{-\beta(1+d_2)T}} \cap \Theta_{\tilde{\gamma}_2}^{\epsilon_0 e^{-\beta(1+d_2)T}} = \emptyset.$$

Proof. Choose $\epsilon_0 = \min\{\frac{\eta_0}{2C_0}, \frac{d_0}{4C_0}\}$. Let $\tilde{\gamma}_k = \tilde{\gamma}_k(t) \in \mathcal{G}(T),\ k = 1, 2$, be two different periodic rays with configurations α_k having lengths p_k, respectively. The rays below are parametrised by the length $t \geq 0$. Let $\tilde{\gamma}_k(t)$ have periods $T_k \leq T,\ k = 1, 2$ and let $\alpha_1 = (i_1, ..., i_{p_1})$. Assume that

$$\Theta_{\tilde{\gamma}_1}^{\epsilon_0 e^{-\beta(1+d_2)T}} \cap \Theta_{\tilde{\gamma}_2}^{\epsilon_0 e^{-\beta(1+d_2)T}} \neq \emptyset. \tag{4.3}$$

Then there exist points $\rho_k = (x_k, \xi_k) \in \mathring{B} \cap \tilde{\gamma}_k,\ k = 1, 2$, and $\rho = (y, \xi) \in \mathring{B}$ such that $\|\rho - \rho_k\| \leq \epsilon_0 e^{-\beta(1+d_2)T},\ k = 1, 2$ and

$$\nu_k(\sigma) = (x_k(\sigma), \xi_k(\sigma)) = (1-\sigma)\rho_k + \sigma\rho \in \mathring{B},\ \sigma \in [0, 1],\ k = 1, 2.$$

Assume that x_1 lies on the segment connecting $u_{i_{p_1}} \in \partial D_{i_{p_1}}$ and $u_{i_1} \in \partial D_{i_1}$, while x_2 lies on the segment connecting $w_{j_{p_2}} \in \partial D_{j_{p_2}}$ and $w_{j_1} \in \partial D_{j_1}$. If $i_1 = j_1$, since $\alpha_1 \neq \alpha_2$, there exist $i_n, i_m \in \{1, ..., r\},\ i_n \neq i_m$, such that the ray $\tilde{\gamma}_1(t)$ issued from ρ_1 follows a configuration $\beta_1 = (i_1, ..., i_{n-1}, i_n),\ 2 \leq n \leq p_1$, while the ray $\tilde{\gamma}_2(t)$ issued form ρ_2 follows a configuration $\beta_2 = (i_1, ..., i_{n-1}, i_m)$. More precisely, i_n and i_m are the first indices in the configurations β_1, β_2, where we have difference. If $i_1 \neq j_1$ we have configurations $\beta_1 = (i_1, ...)$, $\beta_2 = (j_1, ...)$. This case can be covered by the same argument since we prove that $\tilde{\gamma}_\omega(t)$ defined below follows the configurations β_1 and β_2.We omit the details.

Without loss of generality we may assume that β_1, β_2 have lengths less or equal to p_1, that is $n \leq p_1$. Indeed, if

$$\beta_1 = (\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}, i_1, ..., i_{n-1}, i_n),\ \beta_2 = (\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}, i_1, ..., i_{n-1}, i_m),\ n \leq p_1,$$

we may cancel $\underbrace{\alpha_1, ..., \alpha_1}_{\text{k times}}$.

For σ small enough the rays $\tilde{\gamma}_\sigma(t)$, $t \geq 0$, issued from $\nu_1(\sigma)$ will follow the configuration β_1 with reflections on $\partial D_{i_1}, ..., \partial D_{i_n}$, and the ray $\tilde{\zeta}_\sigma(t)$, $t \geq 0$, issued from $\mu_1(\sigma) = (x_1(\sigma), -\xi_1(\sigma)) \in \mathring{B}$ follows a configuration $\bar{\beta}_1 = (i_{p_1}, ...)$. For σ small the n successive reflections of $\tilde{\gamma}_\sigma(t)$ belong to $\mathcal{D}_{\rm out,0}$, as well as the reflection of $\tilde{\zeta}_\sigma(t)$ on $\partial D_{i_{p_1}}$. In general, the rays $\tilde{\gamma}_\sigma(t)$ are not periodic, so after successive reflexions on $\partial D_{i_1}, ..., \partial D_{i_n}$ they may have other reflexions or glancing points and also they may escape to infinity.

Let $0 \leq t \leq d_2 T$ and assume that $\phi_t(v_1(\sigma)) \in \mathring{B}$ for $0 \leq \sigma_0 \leq \sigma \leq \sigma_1 \leq 1$ has the form (4.1). Therefore

$$\|\phi_t(v_1(\sigma_0)) - \phi_t(v_1(\sigma_1))\| = \|\int_{\sigma_0}^{\sigma_1} \frac{d}{ds}(\phi_t(v_1(\sigma))d\sigma\| \leq C_0 e^{\beta t}\|v_1(\sigma_0) - v_1(\sigma_1)\|$$

$$\leq C_0 e^{\beta d_2 T}\|\rho_1 - \rho\| \leq \min\{\frac{\eta_0}{2}, \frac{d_0}{4}\} e^{-\beta T}, \tag{4.4}$$

where we have used (4.2). Let

$$\omega = \max\{\sigma \in [0,1] : \tilde{\gamma}_\sigma(t) \text{ does not follow } \beta_1$$
$$\text{with reflections on } \partial D_{i_1}, ..., \partial D_{i_n} \text{ which belong to } \mathcal{D}_{\rm out,0}$$
$$\text{or } \tilde{\zeta}_\sigma(t) \text{ has not a reflection on } \partial D_{i_{p_1}} \text{ which is in } \mathcal{D}_{\rm out,0}.\}$$

For the rays $\tilde{\gamma}_\omega(t)$, $\tilde{\zeta}_\omega(t)$ there are several cases.

(a1). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ..., i_s)$, $2 \leq s \leq n$, with reflections on $\partial D_{i_1}, ..., \partial D_{i_{s-1}}$ (on ∂D_{i_1} if $s = 2$) and tangency on ∂D_{i_s}.

(a2). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ..., i_{s-1}, i_q)$, $2 \leq s \leq n$, $q \neq s$ with reflections on $\partial D_{i_1}, ..., \partial D_{i_{s-1}}$, and reflection or tangency on ∂D_{i_q}.

(a3). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ..., i_{s-1})$, $2 \leq s \leq n$, with reflections on $\partial D_{i_1}, ..., \partial D_{i_{s-1}}$. After reflection on $\partial D_{i_{s-1}}$ the ray $\tilde{\gamma}_\omega(t)$ does not meet ∂D and it goes to infinity.

In the cases (a1)-(a3) we have $\beta_1 = (i_1, ..., i_s, ...)$.

(b1). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_1, ...)$ with tangency on ∂D_{i_1}.

(b2). $\tilde{\gamma}_\omega(t)$ follows a configuration $\zeta = (i_q, ...)$, $q \neq 1$, $q \neq p_1$ with reflection or tangency on ∂D_{i_q}.

(b3). $\tilde{\gamma}_\omega(t)$ does not meet ∂D and it goes to infinity.

(c1). $\tilde{\gamma}_\omega(t)$ follows β_1, while $\tilde{\zeta}_\omega(t)$ follows a configuration $\bar{\beta}_1 = (i_{p_1}, ...)$ with tangency on $\partial D_{i_{p_1}}$.

(c2). $\tilde{\gamma}_\omega(t)$ follows β_1, while $\tilde{\zeta}_\omega(t)$ follows a configuration $\zeta = (i_{q_1}, ...)$, $q_1 \neq p_1$, $q_1 \neq i_1$ with reflection or tangency on ∂D_{i_q}.

(c3). $\tilde{\gamma}_\omega(t)$ follows β_1, while $\tilde{\zeta}_\omega(t)$ does not meet ∂D and it goes to infinity.

We will show that the cases (a1) - (c3) lead to contradiction.

(a1). Let $\tilde{\gamma}_\omega(t)$ have a tangency at $v_{i_s} \in \partial D_{i_s} \times \mathbb{S}^{d-1}$ for time t_ω. The rays $\tilde{\gamma}_\sigma(t)$ with $0 < \sigma < \omega$ have reflections which belong to $(\partial D_{i_s} \times \mathbb{S}^{d-1}) \cap D_{\text{out},0}$ for $t_{s,\sigma}$ and $t_{s,\sigma} \to t_\omega$ as $\sigma \to \omega$. By continuity, we obtain $v_{i_s} \in \mathcal{D}_{\text{out},0}$ which yields a contradiction with the tangency of v_{i_s}.

(a2). Let $\tilde{\gamma}_\omega(t)$ have reflection at $v_{i_{s-1}} \in \partial D_{i_{s-1}}$ for time t_{s-1} and reflection or tangency at $v_{i_q} \in \partial D_{i_q}$ for time t_q. Let $\tilde{\gamma}_\sigma(t)$ with $0 < \sigma < \omega$ have reflections at $w_{i_{s-1},\sigma} \in \partial D_{i_{s-1}}$ and $w_{i_s,\sigma} \in \partial D_{i_s}$ for times $t_{s-1,\sigma}$ and $t_{s,\sigma}$, respectively. For σ close to ω by (4.2) we deduce that $t_{s-1,\sigma}$ is close to t_{s-1}. This implies $t_q > t_{s-1,\sigma}$ for small σ. We fix $0 < \sigma < \omega$ with this property. Notice that

$$t_q \leq s d_1 \leq \frac{2d_1}{d_0} T_1 \leq d_2 T.$$

Similarly, $t_{s,\sigma} \leq d_2 T$. There are two possibilities: (I). $t_{s,\sigma} < t_q$, (II). $t_{s,\sigma} \geq t_q$. In the case (I), we apply (4.4) with $t = t_{s,\sigma}, \sigma_0 = \sigma, \sigma_1 = \omega$ and obtain

$$\|\pi(\phi_{t_{s,\sigma}}(v_1(\omega))) - w_{i_s,\sigma}\| \leq \frac{\eta_0}{2}.$$

On the other hand, $\pi(\phi_{t_{s,\sigma}}(v_1(\omega)))$ lies on the segment connecting $v_{i_{s-1}}$ and v_{i_q}. Hence this point belongs to $ch(\bigcup_{j \neq s} D_j)$ and the above inequality implies a contradiction.

Passing to the case (II), first suppose that $\tilde{\gamma}_\omega(t)$ has a reflection at v_{i_q}. We apply (4.4) with $t = t_q, \sigma_0 = \sigma, \sigma_1 = \omega$ and deduce

$$\|\pi(\phi_{t_q}(v_1(\sigma))) - v_{i_q}\| \leq \frac{\eta_0}{2}.$$

Since $\pi(\phi_{t_q}(v_1(\sigma)))$ lies on the segment connecting $w_{s-1,\sigma}$ and $w_{s,\sigma}$, we obtain again a contradiction because $v_{i_q} \notin ch(\bigcup_{j \neq q} D_j)$. Now suppose that $\tilde{\gamma}_\omega(t)$ has a tangency at v_{i_q}. Then for sufficiently small $\epsilon > 0$ we have $t_{s,\sigma} > t_q - \epsilon$ and $v_{q,\epsilon} = \pi(\tilde{\gamma}_\omega(t_q - \epsilon)) \in \mathring{B}$. Moreover, for small ϵ we have dist $\Big(v_{q,\epsilon}, ch(\bigcup_{j\neq q} D_j)\Big) > \frac{\eta_0}{2}$. We repeat the above argument applying (4.4) with $t = t_q - \epsilon$, and obtain a contradiction.

(a3). We use the notations in (a1) and (a2). For $t > t_{s-1}$ the ray $\pi(\tilde{\gamma}_\omega(t))$ does not meet D and for $t > t_{s-1} + \epsilon_0 > t_{s-1}$ we have dist $(\pi(\tilde{\gamma}_\omega(t)), D) \geq \epsilon_1 > 0$. Since $t_{s-1,\sigma}$ is close to t_{s-1} for σ close to ω, we have $t_{s,\sigma} \geq t_{s-1,\sigma} + d_0/2 > t_{s-1} + \epsilon_0$ choosing $0 < \epsilon_0 < \frac{d_0}{4}$ and σ sufficiently close to ω. As above, we obtain $t_{s,\sigma} \leq d_2 T_1$. Now we apply (4.4) with $t = t_{s,\sigma}, \sigma_0 = \sigma, \sigma_1 = \omega$ and obtain

$$\|\pi(\phi_{t_{s,\sigma}}(v_1(\sigma))) - \pi(\phi_{t_{s,\sigma}}(v_1(\omega)))\| \leq C_0 e^{\beta d_2 T} \|v_1(\sigma) - v_1(\omega)\|.$$

Taking σ sufficiently close to ω, the right hand side of the above inequality will be less than ϵ_1 and we obtain a contradiction with

$$\text{dist}\,(\pi(\tilde{\gamma}_\omega(t_{s,\sigma})), D) \geq \epsilon_1 > 0.$$

(b1). We repeat the argument of (a1) by using the fact that the rays $\tilde{\gamma}_\sigma(t)$ with $0 < \sigma < \omega$ have reflections which belong to $(\partial D_{i_1} \times \mathbb{S}^{d-1}) \cap D_{\text{out},0}$.

(b2). Let $\tilde{\gamma}_\omega(t)$ have a refection or tangency at $v_{i_q} \in \partial D_{i_q}$ for time t_q and let $\tilde{\gamma}_\sigma(t)$, $0 < \sigma < \omega$ have a refection at $w_{i_1,\sigma} \in \partial D_{i_1}$ for time $t_{i_1,\sigma}$. Let $\tilde{\zeta}_\sigma(t)$ have reflection at $w_{i_{p_1},\sigma} \in \partial D_{i_{p_1}}$. For σ sufficiently small, we have $t_q < t_{i_1,\sigma}$. Indeed, if $t_q \geq t_{i_1,\sigma}$, then the ray $\pi(\tilde{\gamma}_\sigma(t))$ for time $0 < t \leq t_q$ lies in the complement $\mathbb{R}^d \setminus D_{i_1}$. This is impossible because $\pi(\tilde{\gamma}_\sigma(t))$ has a reflection for $t = t_{i_1,\sigma}$. We fix $0 < \sigma < \omega$ with this property. Suppose that $\tilde{\gamma}_\omega(t)$ has a reflection at v_{i_q}. Applying (4.4) with $t = t_q, \sigma_0 = \sigma, \sigma_1 = \omega$, we obtain $\|\pi(\tilde{\gamma}_\sigma(t_q)) - v_{i_q}\| \leq \eta_0/2$. On the other hand, $\pi(\tilde{\gamma}_\sigma(t_q))$ belongs to the segment connecting $w_{i_1,\sigma}$ and $w_{i_{p_1},\sigma}$ which lies in $ch(\bigcup_{j\neq q} D_j)$ and we obtain a contradiction. In the case, when $\tilde{\gamma}_\omega(t)$ has a tangency at v_{i_q} we consider a point $\tilde{\gamma}_\omega(t_q - \epsilon) \in \mathring{B}$ with sufficiently small $\epsilon > 0$ and $t_q - \epsilon < t_{i_1,\sigma}$. For small ϵ we will have dist $\Big(\pi(\tilde{\gamma}_\omega(\mathrm{t_q} - \epsilon)), \mathrm{ch}(\bigcup_{\mathrm{j}\neq \mathrm{q}} \mathrm{D_j})\Big) \geq \frac{2\eta_0}{3}$. We apply (4.4) with $t = t_q - \epsilon$ and one obtains again a contradiction.

(b3). We use the fact that $\tilde{\gamma}_\sigma(t)$, $0 < \sigma < \omega$, has a refection on ∂D_{i_1} and repeat the argument of (a3).

(c1). The rays $\tilde{\zeta}_\sigma(t)$, $0 < \sigma < \omega$, have reflections which belong to $(\partial D_{i_{p_1}} \times \mathbb{S}^{d-1}) \cap D_{\text{out},0}$ and we are in the situation treated in (a1). We repeat the argument of (a1) to obtain a contradiction.

(c2). This case is similar to (b2) and can be treated by the same argument. We omit the details.

(c3). This case is similar to the cases (a3) and (b3) and can be covered by a similar argument. We omit the details.

Finally, notice that by continuity the reflections of the ray $\tilde{\gamma}_\omega(t)$ on $\partial D_{i_1}, ..., \partial D_{i_s}$ and that of $\tilde{\zeta}_\omega(t)$ on $\partial D_{i_{p_1}}$ are in $\mathcal{D}_{\text{out},0}$.

Combining the above cases, we deduce that the existence of $0 < \omega \leq 1$ with the above property is impossible. Thus we conclude that the ray $\tilde{\gamma}_\rho(t)$ issued from ρ follows the configuration β_1. We repeat the above argument for the periodic ray $\tilde{\gamma}_2(t)$ issued from ρ_2 and deduce that $\tilde{\gamma}_\rho(t)$ follows the configuration β_2. Since $\beta_1 \neq \beta_2$, this implies a contradiction with the assumption (4.3). □

Corollary 4.1. *Let $\tilde{\delta}_1$, $\tilde{\delta}_2$ be two periodic primitive rays with periods $T_k \leq T$, $k = 1, 2$, passing through points $\rho_k = (x, \xi_k) \in B, k = 1, 2$. Let (x_k, v_k) be the outgoing representative of ρ_k. The we have*

$$\|v_1 - v_2\| \geq \epsilon_0 e^{-\beta(1+d_2)T}. \tag{4.5}$$

If $x \notin \partial D$, we take (x, ξ_k) as outgoing representative.

Proof. If $x \notin \partial D$, the statement is a trivial consequence of Theorem 4.1. If $x \in \partial D$, consider points $y_k \in \pi(\delta_k)$ in $\mathbb{R}^d \setminus D$ with $\|y_k - x\| = \eta < \min\{1, \frac{d_0}{2}\}$. Assume that

$$\|v_1 - v_2\| < \epsilon_0 e^{-\beta(1+d_2)T}.$$

Then $\|y_1 - y_2\| = 2\eta \sin \frac{\psi}{2}$, where ψ is the angle between the directions $v_1 \in \mathbb{S}^{d-1}$ and $v_2 \in \mathbb{S}^{d-1}$. Clearly,

$$\|v_1 - v_2\|^2 = 2(1 - \cos\psi) = 4\sin^2 \frac{\psi}{2}.$$

For the points $\rho_k = (y_k, v_k) \in \mathring{B}$ we deduce

$$\|\rho_1 - \rho_2\| \leq \sqrt{1 + \eta^2}\|v_1 - v_2\| \leq \epsilon_0 \sqrt{1 + \eta^2} e^{-\beta(1+d_2)T}.$$

and for $\rho = \frac{\rho_1 + \rho_2}{2} \in \mathring{B}$, this implies

$$\|\rho - \rho_k\| < \epsilon_0 e^{-\beta(1+d_2)T},\ k = 1, 2.$$

Applying Theorem 4.1, we obtain a contradiction. □

5. Open problem

The statement of Theorem 4.1 is true for obstacles satisfying (1.1). To apply a perturbation arguments it is important to know that for every $\tilde{\gamma} \in \mathcal{G}(T)$ with $T \geq T_0$ and $x \in \pi(\tilde{\gamma}) \cap \partial D$ there exist $\alpha \gg 1$, $T_0 \gg 1$ and a neighbourhood

$$B(x, e^{-\alpha T}) = \{y \in \partial D : \|x - y\| \leq e^{-\alpha T}\}$$

such that

$$\forall \zeta \in \mathcal{G}(T) \setminus \tilde{\gamma},\ B(x, e^{-\alpha T}) \cap \zeta = \emptyset. \tag{5.1}$$

In general this is not true since there are different periodic rays passing through a point $x \in \partial D$ with different directions (see [PS17, Section 2.1] for examples). On the other hand, in [PS17, Theorem 6.2.3] it was established that for generic obstacles for every $x \in \partial D$ there exists at most *one direction* $\xi \in \mathbb{S}^{d-1}$ (up to symmetry with respect to the normal to ∂D at x) such that (x, ξ) could generate a periodic ray. The reader may consult [PS17, Section 6.2]) for the precise definition of generic obstacles. Since there are only finite number periodic rays with period T, for generic obstacles every point $x \in \partial D$ has a suitably small neighbourhood with the property mentioned above. However, the size of these neighbourhoods could be extremely small and their dependence of T is unknown. We conjecture that there exist $\alpha \gg 1, T_0 \gg 1$, such that for generic obstacles for all $\zeta \in \mathcal{G}(T)\setminus\tilde{\gamma}, T \geq T_0$ the property (5.1) holds. For metrics on compact Riemannian manifolds with negative curvature a relation similar to (5.1) has been proved in [Sch20, Proposition 4]) without a generic assumption.

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